

Late-time decay of coupled electromagnetic and gravitational perturbations outside an extremal charged black hole

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In this paper we employ the results of a previous paper on the late-time decay of scalar-field perturbations of an extreme Reissner-Nordstrom black hole, in order to find the late-time decay of coupled electromagnetic and gravitational perturbations of this black hole. We explicitly write the late-time tails of Moncrief's gauge invariant variables and of the perturbations of the metric tensor and the electromagnetic field tensor in the Regge-Wheeler gauge. We discuss some of the consequences of the results and relations to previous works.

I. INTRODUCTION

Coupled electromagnetic and gravitational perturbations of the Reissner-Nordstrom black hole have been studied by various authors, and several formalisms were developed for studying them [1-11]. In Refs. [2-4], Moncrief found, using the Hamiltonian formulation of the Einstein-Maxwell equations, gauge invariant variables (under both electromagnetic gauge transformations and infinitesimal diffeomorphisms) which can be used as the perturbation variables. In particular, one can use these gauge invariant variables in order to express all metric and electromagnetic perturbations after specifying the gauge. In Ref. [9], Bicak related the various formalisms and explicitly wrote all metric and electromagnetic perturbations in terms of Moncrief's gauge invariant variables in the Regge-Wheeler gauge.

Later, in [10], Bicak employed his results from [9] and showed that scalar-field perturbations serve as a prototype for the coupled electromagnetic and gravitational perturbations; he then used his results from Ref. [12] that analyzed scalar-field perturbations to deduce the late-time behavior of the coupled perturbations. In both cases (scalar and coupled perturbations), Bicak obtained a late-time asymptotic behavior that corresponds to a power law decay¹. This behavior of asymptotic power law decay of perturbations also appears in the simpler case of a Schwarzschild black hole (see, for example, Refs. [14-15]).

In a recent paper [13], we revisited Bicak's analysis [12] of the scalar-field perturbations, concentrating on the *extremal* Reissner-Nordstrom (ERN) black hole. In this case, we found that the late-time asymptotic behavior of the scalar perturbations is again of the form of a power law decay, but with a different exponent compared with the one obtained by Bicak (for the same form of initial data). The power law that we found is exactly the same as the one obtained in [11] using both numerical computations (that correspond to some specific cases) and analytical derivations.

In this paper, we use the results of [13] to find the late-time decay of coupled electromagnetic and gravitational perturbations in ERN geometry, in complete analogy with the derivations of Bicak's [10]. In other words, we use our understanding of the scalar-field late-time decay from [13], and the fact that scalar-field perturbations serve as a prototype for the coupled perturbations [10], to find the late-time decay of all the relevant quantities describing coupled perturbations in ERN spacetime. Moreover, we examine how this decay changes under different choices of initial data. In this, we revisit the analysis of [10] and develop it further.

The organization of this paper is as follows. In Sec. II, we introduce the decoupled wave equations satisfied by certain combinations Ψ_{\pm} of Moncrief's gauge invariant variables, and the close resemblance they have to the scalar-field wave equation. We then deduce, as in [10], that scalar-field perturbations serve as a prototype for coupled perturbations. In Sec. III, we describe the various kinds of initial data we can consider for the perturbations, and classify them. Later, in Sec. IV, we use the results of the previous paper [13] to find the late-time tails of Ψ_{\pm} corresponding to different choices of initial data. We then readily determine the late-time tails of Moncrief's gauge invariant variables in Sec. V using the definitions of the combinations Ψ_{\pm} . In Sec. VI, we employ relations from Ref. [9] and find the late-time decay of the perturbations of the metric tensor and the electromagnetic field tensor in the Regge-Wheeler gauge. We conclude in Sec. VII.

¹ In this statement, we exclude perturbations that correspond to a slowly rotating Kerr-Newman black hole.

II. SCALAR-FIELD PERTURBATIONS AS A PROTOTYPE FOR COUPLED PERTURBATIONS

The ERN geometry is given in Schwarzschild-like coordinates by the line element

$$ds^2 = -(1 - M/r)^2 dt^2 + (1 - M/r)^{-2} dr^2 + r^2 d\Omega^2,$$

where M is the mass of the black hole and $d\Omega^2 = d\theta^2 + \sin^2(\theta) d\phi^2$. In this paper, we focus on the external domain, $r > M$. Throughout the analysis, we use the “tortoise coordinate”, $r_*(r)$, defined in the usual way by $dr/dr_* = (1 - M/r)^2$. Fixing the integration constant by setting $r_*(2M) = 0$, we get

$$r_*(r) = r - M - \frac{M^2}{r - M} + 2M \ln\left(\frac{r}{M} - 1\right).$$

This function diverges to $+\infty$ at $r \rightarrow \infty$, to $-\infty$ at $r \rightarrow M$, and vanishes at $r = 2M$. At the asymptotic regions $r \rightarrow \infty$ and $r \rightarrow M$, we can find the inverse function $r(r_*)$ iteratively,

$$r \sim r_* - 2M \ln\left(\frac{r_*}{M}\right) \quad , \quad r_* \rightarrow \infty, \quad (2.1)$$

and

$$r \sim M + \frac{M^2}{|r_*|} \left[1 + \frac{2M}{|r_*|} \ln\left(\frac{|r_*|}{M}\right) \right] \quad , \quad r_* \rightarrow -\infty. \quad (2.2)$$

When considering coupled electromagnetic and gravitational perturbations of ERN black hole, one exploits the spherical symmetry of the background and expands the perturbations in scalar, vector, and tensor harmonics. Then, one identifies the corresponding parity (under inversion transformation) and distinguishes between even and odd parity perturbations. This way, for each l in the harmonic expansion, we have even and odd parity perturbations (and the corresponding perturbation equations for the two types of parity decouple) [1-11].

In Moncrief’s gauge invariant formalism [2-4], the $l = 1$ perturbations are fully determined by a gauge invariant function $P_f(r, t)$ in the odd parity case and by a gauge invariant function $H(r, t)$ in the even parity case. For $l \geq 2$, the odd parity perturbations are determined by two gauge invariant functions, $\hat{\pi}_f(r, t)$ and $\hat{\pi}_g(r, t)$, and the even parity perturbations also by two gauge invariant functions, $H(r, t)$ and $Q(r, t)$.²

Introducing the standard combinations of Moncrief’s gauge invariant functions (for $l \geq 2$) [2-4,9-11],

$$P_{\pm} = (2\sigma)^{-1/2} \left[\pm (\sigma \pm 3M)^{1/2} \hat{\pi}_f + (\sigma \mp 3M)^{1/2} \hat{\pi}_g \right] \quad (2.3)$$

and

$$R_{\pm} = (2\sigma)^{-1/2} \left[(\sigma \pm 3M)^{1/2} H \mp (\sigma \mp 3M)^{1/2} Q \right], \quad (2.4)$$

where (in the extremal case)

$$\sigma = M(2l + 1), \quad (2.5)$$

we use the conventional notations Ψ_{\pm} to denote P_{\pm} in the case of odd perturbations and R_{\pm} in the case of even perturbations. For $l = 1$, Ψ_+ denotes P_f and H for odd and even perturbations, respectively. Ψ_- has no meaning for $l = 1$.

Now, we can describe the dynamics of all the coupled electromagnetic and gravitational perturbations of an ERN black hole by the following set of two decoupled wave equations satisfied by Ψ_{\pm} [9, 10]:

$$\Psi_{\pm,tt} - \Psi_{\pm,r_*r_*} + V_{l\pm}^{\text{odd,even}}(r_*) \Psi_{\pm} = 0, \quad (2.6)$$

where the effective potentials $V_{l\pm}^{\text{odd,even}}$ are given in the extremal case (ERN) by [9, 10]

$$V_{l\pm}^{\text{odd}} = \frac{1}{r^2} \left(1 - \frac{M}{r} \right)^2 \left(L - \frac{3M}{r} + \frac{4M^2}{r^2} \pm \frac{\sigma}{r} \right)$$

² Throughout the analysis, our notations of the various functions are similar to those used by Bicak in [9, 10], and are very close to the notations used by Moncrief in [2-4].

and

$$V_{l\pm}^{\text{even}} = \left(1 - \frac{M}{r}\right)^2 (V \pm \sigma S),$$

where

$$\begin{aligned} S &= \frac{1}{(r\Lambda)^2} \left[\frac{L^2 - 4}{r} + \frac{12M}{r^2} \left(1 - \frac{M}{r} + \frac{M^2}{3r^2}\right) \right], \\ V &= \frac{1}{(r\Lambda)^2} \left\{ (L-2) \left[L(L-2) + 3(3L-2) \frac{M}{r} - \frac{4M^2}{r^2} \left(L - 4 + \frac{16M}{r} - \frac{6M^2}{r^2} \right) \right] \right. \\ &\quad \left. + \frac{4}{r^2} \left[9M^2 \left(L - 1 + \frac{M}{r} \right) - \frac{M^2}{r^2} \left(\frac{8M^4}{r^2} - \frac{32M^3}{r} + 39M^2 \right) \right] \right\}, \\ L &= l(l+1) \quad , \quad \Lambda = L - 2 \left(1 - \frac{M}{r} \right) \left(1 - \frac{2M}{r} \right), \end{aligned}$$

and σ is given by Eq. (2.5). Note that for $l = 1$, Eq. (2.6) is meaningful only for Ψ_+ . As mentioned above, all the electromagnetic and gravitational perturbations can be obtained once all the Ψ_{\pm} are known.

As shown in [10], the effective potentials $V_{l\pm}^{\text{odd,even}}$ have the same qualitative properties as the scalar-field effective potential [10, 13],

$$F_l^{\text{scalar}} = \frac{1}{r^2} \left(1 - \frac{M}{r} \right)^2 \left(L + \frac{2M}{r} - \frac{2M^2}{r^2} \right).$$

In addition, the potentials $V_{l\pm}^{\text{odd,even}}$ have the following asymptotic behaviors near spatial infinity and near the horizon:

$$V_{l\pm}^{\text{odd,even}} = \frac{l(l+1)}{r^2} + \mathcal{O}\left(\frac{M}{r^3}\right) \quad , \quad r \rightarrow \infty$$

and

$$V_{l+}^{\text{odd,even}} = (l+1)(l+2) \left(\frac{r-M}{M^2} \right)^2 + \mathcal{O} \left[M^4 \left(\frac{r-M}{M^2} \right)^3 \right] \quad , \quad r \rightarrow M,$$

$$V_{l-}^{\text{odd,even}} = (l-1)l \left(\frac{r-M}{M^2} \right)^2 + \mathcal{O} \left[M^4 \left(\frac{r-M}{M^2} \right)^3 \right] \quad , \quad r \rightarrow M.$$

With the help of Eqs. (2.1) and (2.2), we can write these asymptotic behaviors in terms of r_* ,

$$V_{l\pm}^{\text{odd,even}} = \frac{l(l+1)}{r_*^2} + 4M \frac{l(l+1)}{|r_*|^3} \ln(|r_*|/M) + \mathcal{O}(Mr_*^{-3}) \quad , \quad r_* \rightarrow \infty \quad (2.7)$$

and

$$V_{l+}^{\text{odd,even}} = \frac{(l+1)(l+2)}{r_*^2} + 4M \frac{(l+1)(l+2)}{|r_*|^3} \ln(|r_*|/M) + \mathcal{O}(Mr_*^{-3}) \quad , \quad r_* \rightarrow -\infty, \quad (2.8)$$

$$V_{l-}^{\text{odd,even}} = \frac{(l-1)l}{r_*^2} + 4M \frac{(l-1)l}{|r_*|^3} \ln(|r_*|/M) + \mathcal{O}(Mr_*^{-3}) \quad , \quad r_* \rightarrow -\infty. \quad (2.9)$$

In these asymptotic regions, the scalar-field potential F_l^{scalar} takes the asymptotic forms

$$F_l^{\text{scalar}} = \frac{l(l+1)}{r^2} + \mathcal{O}\left(\frac{M}{r^3}\right) \quad , \quad r \rightarrow \infty$$

and

$$F_l^{\text{scalar}} = l(l+1) \left(\frac{r-M}{M^2} \right)^2 + \mathcal{O} \left[M^4 \left(\frac{r-M}{M^2} \right)^3 \right] \quad , \quad r \rightarrow M.$$

Using Eqs. (2.1) and (2.2) as before, we can write these in terms of r_* ,

$$F_l^{\text{scalar}} = \frac{l(l+1)}{r_*^2} + 4M \frac{l(l+1)}{|r_*|^3} \ln(|r_*|/M) + \mathcal{O}(Mr_*^{-3}) \quad , \quad r_* \rightarrow \pm\infty. \quad (2.10)$$

It is now clear from Eqs. (2.7) and (2.10) that the leading and the next-to-leading (in r_*^{-1}) order terms in the potentials F_l^{scalar} , $V_{l+}^{\text{odd,even}}$, and $V_{l-}^{\text{odd,even}}$ at the limit $r_* \rightarrow \infty$ are the same. That is, the centrifugal potential term and the leading curvature-induced term that appear in these potentials at the limit $r_* \rightarrow \infty$ are the same. Therefore, at spatial infinity, these potentials satisfy [to leading and next-to-leading (in r_*^{-1}) order]

$$V_{l\pm}^{\text{odd,even}} = F_l^{\text{scalar}} \quad , \quad r_* \rightarrow +\infty. \quad (2.11)$$

Analogously, from Eqs. (2.8), (2.9), and (2.10), one can readily see that near the horizon ($r_* \rightarrow -\infty$), these potentials satisfy [to leading and next-to-leading (in r_*^{-1}) order]

$$V_{l\pm}^{\text{odd,even}} = F_{l\pm 1}^{\text{scalar}} \quad , \quad r_* \rightarrow -\infty. \quad (2.12)$$

The reason for keeping the next-to-leading order terms (curvature-induced terms) in the asymptotic expansions of the potentials is that these terms are essential for determining the late-time decay of perturbations with initial data that is of compact support and is well separated from the horizon (see below for more details).

As shown and discussed in Refs. [10, 11], for regular initial data, we get that Ψ_{\pm} are regular at the horizon and at future null infinity (FNI).

Therefore, Ψ_{\pm} satisfy wave equations with effective potentials that have the same asymptotic forms as the scalar-field potential and have regular boundary conditions. Now, since these properties are the only ones we needed in [13] in order to determine the late-time decay of the scalar perturbations, we can use our results and experience from [13] to determine the late-time decay of Ψ_{\pm} . Then, as a result, we get the late-time behavior of the coupled perturbations.

III. INITIAL-VALUE SETUP

When analyzing the dynamics (and, in particular, the late-time decay) of perturbations, a key ingredient is the specification of initial-value data. We consider characteristic initial-value problem for Ψ_{\pm} , just like in [13], for which the initial value of the perturbations is specified along two intersecting radial null rays, $u = \text{const}$ and $v = \text{const}$, where u and v are the usual null coordinates, $u = t - r_*$ and $v = t + r_*$.

According to Refs. [11, 13, 16], the various forms of late-time decay of *scalar* perturbations of an ERN black hole can be classified according to the values of the Aretakis and Newman-Penrose (NP) constants associated with their initial-value data. If either the Aretakis constant or the NP constant is nonzero, the late-time decay would be $\sim t^{-(2l+2)}$; otherwise, it would be $\sim t^{-(2l+3)}$. Now, since we have (for $l \geq 2$) four types of gauge invariant combinations Ψ_{\pm} (two combinations \pm for each parity) that behave as scalar fields (at least as long as their late-time decay is concerned) and four types of initial-value data for scalar perturbations (according to whether their Aretakis and NP constants vanish), we have a total of $4^4 = 256$ scenarios.³ Since this number is very large, we shall only consider the four scenarios that make no difference between the initial data of the various combinations and parities. In other words, we consider all the different Ψ_{\pm} on an equal footing with respect to the type of initial data. The four scenarios result from the four types of initial-value data, defined as follows.

Type A. “Horizon-based initial data” – Initial data for which Ψ_{\pm} have nonvanishing Aretakis constants and vanishing NP constants. That is, we consider initial data with generic regular behavior across the horizon.

Type B. “FNI-based initial data” – Initial data for which Ψ_{\pm} have vanishing Aretakis constants and nonvanishing NP constants.

³ For $l = 1$, only Ψ_+ is meaningful and therefore we have $4^2 = 16$ scenarios.

Type C. Initial data for which Ψ_{\pm} have nonvanishing Aretakis constants and nonvanishing NP constants. This type is essentially a combination of the two types A and B.

Type D. Initial data for which Ψ_{\pm} have vanishing Aretakis and NP constants.

Note that since the asymptotic form of the effective potentials $V_{l\pm}^{\text{odd,even}}$ is different in the two limits $r_* \rightarrow \infty$ and $r_* \rightarrow -\infty$ [cf. Eqs. (2.10), (2.11), and (2.12)], the late-time decays of Ψ_{\pm} that result from the two types of initial data A and B are generally different (as opposed to scalar perturbations, where the decay is the same for all three types of initial data A, B, and C).

Note also that for initial data of types A and C, Ψ_{\pm} that correspond to a certain l value have nonvanishing $l \pm 1$ Aretakis constants.

We can now turn to calculate the late-time tails of Ψ_{\pm} that correspond to the four types of initial-value data A-D.

IV. LATE-TIME TAILS OF Ψ_{\pm}

As discussed above, we can determine the late-time decay of Ψ_{\pm} using our understanding of scalar-field perturbations. As mentioned, in the case of scalar perturbations of an ERN black hole, if either the Aretakis constant or the NP constant is nonzero, the late-time decay would be $\sim t^{-(2l+2)}$; otherwise, it would be $\sim t^{-(2l+3)}$. As discussed in detail in [13], the $t^{-(2l+2)}$ decay is associated with the centrifugal potential term that appears in the asymptotic form of the scalar-field effective potential [cf. Eq. (2.10)]. Moreover, it is well-known that for initial data with vanishing Aretakis and NP constants, the leading tail $t^{-(2l+3)}$ (for scalar perturbations) is formed from the scattering of the perturbations off the leading, curvature-induced part of the effective potential [see Eq. (2.10) for example of this part of the effective potential]. See Refs. [17-19, 12-14] for further details. Note that this tail is also formed if the initial data have a generic regular behavior across the horizon or FNI (corresponds to types A, B, and C); However, in the scalar case, the tails that result from the centrifugal part of the potential [$\sim t^{-(2l+2)}$] dominates these curvature-induced tails [$\sim t^{-(2l+3)}$]. We will see below that this is not always the case for Ψ_{\pm} .

Now, since Eqs. (2.12) and (2.11) apply to both parts (centrifugal and leading, curvature induced) of the asymptotic effective potential, we can determine the late-time decay of Ψ_{\pm} (that correspond to certain initial data) by considering the two types of contributions coming from the two asymptotic regions $r_* \rightarrow +\infty$ and $r_* \rightarrow -\infty$. Specifically, for each asymptotic region, we begin by considering the two types of contributions (associated with the centrifugal and curvature-induced parts of the potential) to the late-time tails of *scalar-field* perturbations with the same kind of initial data; then, the corresponding contributions to the tails of Ψ_{\pm} are determined according to Eqs. (2.12) and (2.11): The contributions to the tails of Ψ_{\pm} from the region $r_* \rightarrow -\infty$ are the same as the tails of the scalar perturbations, but with a different l value: $l \rightarrow l \pm 1$; and the contributions to the tails of Ψ_{\pm} from the region $r_* \rightarrow +\infty$ are the same as the tails of the scalar perturbations (with the same l value).

Now, after discussing the reasoning, we find the late-time decay of Ψ_{\pm} for the initial data A-D.

A. Type A initial data

Since the NP constants (built from Ψ_{\pm}) vanish, the only contribution from the region $r_* \rightarrow +\infty$ comes from the leading, curvature-induced part of the potential, and therefore, results in a tail $t^{-(2l+3)}$ for both scalar perturbations and Ψ_{\pm} , in accordance with Eq. (2.11). In the region $r_* \rightarrow -\infty$, nonvanishing Aretakis constants, associated with type A initial data, yield the leading *scalar-field* tail $t^{-(2l+2)}$ (that results from the centrifugal part of the potential). The corresponding contributions to the tails of Ψ_{\pm} are $t^{-(2l+4)}$ for Ψ_+ ($l \rightarrow l + 1$) and t^{-2l} for Ψ_- ($l \rightarrow l - 1$), in accordance with Eq. (2.12).

In summary, the leading tail of Ψ_+ is $t^{-(2l+3)}$ (associated with the region $r_* \rightarrow +\infty$) and the leading tail of Ψ_- is t^{-2l} (associated with the region $r_* \rightarrow -\infty$). Here, in the case of Ψ_+ , we see an example where a curvature-induced tail can dominate a tail that results from the centrifugal (“flat space”) part of the potential.

We shall use superscripts to denote the type of initial data that corresponds to the quantity under consideration. We get

$$\Psi_+^{(A)} \sim t^{-(2l+3)} \quad , \quad \Psi_-^{(A)} \sim t^{-2l}.$$

B. Type B initial data

In this case, it is clear that the contribution from the region $r_* \rightarrow -\infty$ to the *scalar-field* tail is $t^{-(2l+3)}$. As a result, the contributions to Ψ_{\pm} are $t^{-(2l+5)}$ for Ψ_+ and $t^{-(2l+1)}$ for Ψ_- .

The leading contribution from the region $r_* \rightarrow \infty$ to the *scalar-field* tail is $t^{-(2l+2)}$, which is also the tail of both Ψ_{\pm} by virtue of Eq. (2.11).

In summary, we get the leading tails

$$\Psi_+^{(B)} \sim t^{-(2l+2)} \quad , \quad \Psi_-^{(B)} \sim t^{-(2l+1)}.$$

C. Type C initial data

In this case, both regions ($r_* \rightarrow \pm\infty$) contribute to the *scalar-field* tail $t^{-(2l+2)}$ (this is the leading tail). The contribution to Ψ_{\pm} from the region $r_* \rightarrow \infty$ is of course the same. The contribution to Ψ_{\pm} from the region $r_* \rightarrow -\infty$ is $t^{-(2l+4)}$ for Ψ_+ and t^{-2l} for Ψ_- .

In summary, we get the leading tails

$$\Psi_+^{(C)} \sim t^{-(2l+2)} \quad , \quad \Psi_-^{(C)} \sim t^{-2l}.$$

D. Type D initial data

Both regions ($r_* \rightarrow \pm\infty$) contribute to the *scalar-field* tail $t^{-(2l+3)}$. The contribution to Ψ_{\pm} from the region $r_* \rightarrow \infty$ is of course the same. The contribution to Ψ_{\pm} from the region $r_* \rightarrow -\infty$ is $t^{-(2l+5)}$ for Ψ_+ and $t^{-(2l+1)}$ for Ψ_- .

In summary, we get the leading tails

$$\Psi_+^{(D)} \sim t^{-(2l+3)} \quad , \quad \Psi_-^{(D)} \sim t^{-(2l+1)}.$$

V. LATE-TIME DECAY OF MONCRIEF'S GAUGE INVARIANT QUANTITIES

Now, after we found the late-time decay of Ψ_{\pm} for initial data A-D, we can find the corresponding late-time tails of Moncrief's gauge invariant quantities. We begin by writing Moncrief's quantities in terms of Ψ_{\pm} (for $l \geq 2$) by solving Eqs. (2.3) and (2.4) for them. For odd parity perturbations, we get ($l \geq 2$)

$$\hat{\pi}_g = (2\sigma)^{-1/2} \left[(\sigma - 3M)^{1/2} P_+ + (\sigma + 3M)^{1/2} P_- \right]$$

and

$$\hat{\pi}_f = (2\sigma)^{-1/2} \left[(\sigma + 3M)^{1/2} P_+ - (\sigma - 3M)^{1/2} P_- \right],$$

and for even parity perturbations, we get ($l \geq 2$)

$$H = (2\sigma)^{-1/2} \left[(\sigma + 3M)^{1/2} R_+ + (\sigma - 3M)^{1/2} R_- \right]$$

and

$$Q = (2\sigma)^{-1/2} \left[-(\sigma - 3M)^{1/2} R_+ + (\sigma + 3M)^{1/2} R_- \right].$$

Note that P_{\pm} and R_{\pm} are just Ψ_{\pm} for odd and even perturbations, respectively. Note that for $l \geq 2$, $\sigma \geq 5M$ [see Eq. (2.5)].

For $l = 1$, Ψ_+ denotes P_f and H for odd and even perturbations, respectively (Ψ_- has no meaning for $l = 1$).

Now, we can readily obtain the late-time decay of Moncrief's quantities by direct substitution of the results from the previous section, and identification of the leading tail.

A. Type A initial data

1. Odd and even parity perturbations with $l \geq 2$

Between the two quantities $\Psi_+^{(A)}$ and $\Psi_-^{(A)}$, the one that decays faster is $\Psi_+^{(A)}$. Therefore, the decay is the same as that of $\Psi_-^{(A)}$, and we get

$$\hat{\pi}_g^{(A)}, \hat{\pi}_f^{(A)}, H^{(A)}, Q^{(A)} \sim t^{-2l}.$$

2. Odd and even parity perturbations with $l = 1$

In these two cases, the decay is simply the same as that of $\Psi_+^{(A)}$ with $l = 1$. Therefore,

$$P_f^{(A)}, H^{(A)} \sim t^{-5}.$$

B. Type B initial data

1. Odd and even parity perturbations with $l \geq 2$

Between $\Psi_+^{(B)}$ and $\Psi_-^{(B)}$, the one that decays faster is $\Psi_+^{(B)}$. Therefore, the decay is the same as that of $\Psi_-^{(B)}$, and we get

$$\hat{\pi}_g^{(B)}, \hat{\pi}_f^{(B)}, H^{(B)}, Q^{(B)} \sim t^{-(2l+1)}.$$

2. Odd and even parity perturbations with $l = 1$

The decay is the same as that of $\Psi_+^{(B)}$ with $l = 1$. Therefore,

$$P_f^{(B)}, H^{(B)} \sim t^{-4}.$$

C. Type C initial data

1. Odd and even parity perturbations with $l \geq 2$

Between $\Psi_+^{(C)}$ and $\Psi_-^{(C)}$, the one that decays faster is $\Psi_+^{(C)}$. Therefore, the decay is the same as that of $\Psi_-^{(C)}$, and we get

$$\hat{\pi}_g^{(C)}, \hat{\pi}_f^{(C)}, H^{(C)}, Q^{(C)} \sim t^{-2l}.$$

2. Odd and even parity perturbations with $l = 1$

The decay is the same as that of $\Psi_+^{(C)}$ with $l = 1$. Therefore,

$$P_f^{(C)}, H^{(C)} \sim t^{-4}.$$

D. Type D initial data

1. Odd and even parity perturbations with $l \geq 2$

Between $\Psi_+^{(D)}$ and $\Psi_-^{(D)}$, the one that decays faster is $\Psi_+^{(D)}$. Therefore, the decay is the same as that of $\Psi_-^{(D)}$, and we get

$$\hat{\pi}_g^{(D)}, \hat{\pi}_f^{(D)}, H^{(D)}, Q^{(D)} \sim t^{-(2l+1)}.$$

2. Odd and even parity perturbations with $l = 1$

The decay is the same as that of $\Psi_+^{(D)}$ with $l = 1$. Therefore,

$$P_f^{(D)}, H^{(D)} \sim t^{-5}.$$

VI. LATE-TIME DECAY OF THE PERTURBATIONS OF THE METRIC TENSOR AND THE ELECTROMAGNETIC FIELD TENSOR

In order to find the late-time tails of coupled perturbations, we employ the results of Ref. [9], where (in Sec. 3) the components of the metric tensor and the electromagnetic field tensor were given in terms of Moncrief's gauge invariant quantities. The gauge used in these expressions is the Regge-Wheeler gauge (for $l \geq 2$). For $l = 1$ perturbations, specific gauges that simplify the expressions further were chosen such that, in addition to the metric perturbation components that vanish for $l \geq 2$ due to the gauge choice, we have $\delta g_{r\phi} = 0$ for odd parity and $\delta g_{\theta\theta} = \delta g_{\phi\phi} = 0$ for even parity.

It is important to note that $l = 1$ odd perturbations include stationary perturbations that correspond to a slowly rotating Kerr-Newman black hole. Specifically, $l = 1$ odd perturbations with $P_f = 0$ (in ERN spacetime) generally yield [see Eqs. (61) and (62) in [9]]

$$\delta g_{t\phi} = - \left(\frac{2M}{r} - \frac{M^2}{r^2} \right) \sin^2(\theta) \delta a \quad , \quad \delta A_\phi = \pm \frac{2M}{r} \sin^2(\theta) \delta a,$$

where δa is a small constant determined by the initial data (and corresponds to the rotation parameter of the black hole), A_μ is the electromagnetic potential, and the \pm sign corresponds to the charge of the black hole ($\pm M$). In this paper, we ignore this kind of perturbations; by $l = 1$ odd perturbations, we mean perturbations beyond the stationary Kerr-Newman ones or perturbations with $\delta a = 0$. These perturbations decay at late time and have a tail.

After substituting the results from the previous section into the expressions from [9] and identifying the leading tail, we get the late-time decay of the coupled perturbations. The results are presented below (all the components that are not written down are either obtained by symmetry or vanish).

A. Type A initial data

1. Odd parity perturbations with $l \geq 2$

$$\delta g_{t\phi}^{(A)} \sim t^{-2l} \quad , \quad \delta g_{r\phi}^{(A)} \sim t^{-(2l+1)},$$

$$\delta F_{t\phi}^{(A)} \sim t^{-(2l+1)} \quad , \quad \delta F_{r\phi}^{(A)} \sim t^{-2l} \quad , \quad \delta F_{\theta\phi}^{(A)} \sim t^{-2l}.$$

2. *Odd parity perturbations with $l = 1$*

$$\delta g_{t\phi,l=1}^{(A)} \sim t^{-5}$$

$$\delta F_{t\phi,l=1}^{(A)} \sim t^{-6} \quad , \quad \delta F_{r\phi,l=1}^{(A)} \sim t^{-5} \quad , \quad \delta F_{\theta\phi,l=1}^{(A)} \sim t^{-5}.$$

3. *Even parity perturbations with $l \geq 2$*

$$\delta g_{tt}^{(A)} \sim t^{-2l} \quad , \quad \delta g_{rr}^{(A)} \sim t^{-2l} \quad , \quad \delta g_{\theta\theta}^{(A)} \sim t^{-2l} \quad , \quad \delta g_{\phi\phi}^{(A)} \sim t^{-2l} \quad , \quad \delta g_{rt}^{(A)} \sim t^{-(2l+1)},$$

$$\delta F_{tr}^{(A)} \sim t^{-2l} \quad , \quad \delta F_{t\theta}^{(A)} \sim t^{-2l} \quad , \quad \delta F_{r\theta}^{(A)} \sim t^{-(2l+1)}.$$

4. *Even parity perturbations with $l = 1$*

$$\delta g_{tt}^{(A)} \sim t^{-5} \quad , \quad \delta g_{rr}^{(A)} \sim t^{-5} \quad , \quad \delta g_{rt}^{(A)} \sim t^{-6},$$

$$\delta F_{tr}^{(A)} \sim t^{-5} \quad , \quad \delta F_{t\theta}^{(A)} \sim t^{-5} \quad , \quad \delta F_{r\theta}^{(A)} \sim t^{-6}.$$

B. Type B initial data

1. *Odd parity perturbations with $l \geq 2$*

$$\delta g_{t\phi}^{(B)} \sim t^{-(2l+1)} \quad , \quad \delta g_{r\phi}^{(B)} \sim t^{-(2l+2)},$$

$$\delta F_{t\phi}^{(B)} \sim t^{-(2l+2)} \quad , \quad \delta F_{r\phi}^{(B)} \sim t^{-(2l+1)} \quad , \quad \delta F_{\theta\phi}^{(B)} \sim t^{-(2l+1)}.$$

2. *Odd parity perturbations with $l = 1$*

$$\delta g_{t\phi,l=1}^{(B)} \sim t^{-4}$$

$$\delta F_{t\phi,l=1}^{(B)} \sim t^{-5} \quad , \quad \delta F_{r\phi,l=1}^{(B)} \sim t^{-4} \quad , \quad \delta F_{\theta\phi,l=1}^{(B)} \sim t^{-4}.$$

3. *Even parity perturbations with $l \geq 2$*

$$\delta g_{tt}^{(B)} \sim t^{-(2l+1)} \quad , \quad \delta g_{rr}^{(B)} \sim t^{-(2l+1)} \quad , \quad \delta g_{\theta\theta}^{(B)} \sim t^{-(2l+1)} \quad , \quad \delta g_{\phi\phi}^{(B)} \sim t^{-(2l+1)} \quad , \quad \delta g_{rt}^{(B)} \sim t^{-(2l+2)},$$

$$\delta F_{tr}^{(B)} \sim t^{-(2l+1)} \quad , \quad \delta F_{t\theta}^{(B)} \sim t^{-(2l+1)} \quad , \quad \delta F_{r\theta}^{(B)} \sim t^{-(2l+2)}.$$

4. *Even parity perturbations with $l = 1$*

$$\begin{aligned} \delta g_{tt}^{(B)} &\sim t^{-4} \quad , \quad \delta g_{rr}^{(B)} \sim t^{-4} \quad , \quad \delta g_{rt}^{(B)} \sim t^{-5}, \\ \delta F_{tr}^{(B)} &\sim t^{-4} \quad , \quad \delta F_{t\theta}^{(B)} \sim t^{-4} \quad , \quad \delta F_{r\theta}^{(B)} \sim t^{-5}. \end{aligned}$$

C. Type C initial data

1. *Odd parity perturbations with $l \geq 2$*

$$\begin{aligned} \delta g_{t\phi}^{(C)} &\sim t^{-2l} \quad , \quad \delta g_{r\phi}^{(C)} \sim t^{-(2l+1)}, \\ \delta F_{t\phi}^{(C)} &\sim t^{-(2l+1)} \quad , \quad \delta F_{r\phi}^{(C)} \sim t^{-2l} \quad , \quad \delta F_{\theta\phi}^{(C)} \sim t^{-2l}. \end{aligned}$$

2. *Odd parity perturbations with $l = 1$*

$$\begin{aligned} \delta g_{t\phi,l=1}^{(C)} &\sim t^{-4} \\ \delta F_{t\phi,l=1}^{(C)} &\sim t^{-5} \quad , \quad \delta F_{r\phi,l=1}^{(C)} \sim t^{-4} \quad , \quad \delta F_{\theta\phi,l=1}^{(C)} \sim t^{-4}. \end{aligned}$$

3. *Even parity perturbations with $l \geq 2$*

$$\begin{aligned} \delta g_{tt}^{(C)} &\sim t^{-2l} \quad , \quad \delta g_{rr}^{(C)} \sim t^{-2l} \quad , \quad \delta g_{\theta\theta}^{(C)} \sim t^{-2l} \quad , \quad \delta g_{\phi\phi}^{(C)} \sim t^{-2l} \quad , \quad \delta g_{rt}^{(C)} \sim t^{-(2l+1)}, \\ \delta F_{tr}^{(C)} &\sim t^{-2l} \quad , \quad \delta F_{t\theta}^{(C)} \sim t^{-2l} \quad , \quad \delta F_{r\theta}^{(C)} \sim t^{-(2l+1)}. \end{aligned}$$

4. *Even parity perturbations with $l = 1$*

$$\begin{aligned} \delta g_{tt}^{(C)} &\sim t^{-4} \quad , \quad \delta g_{rr}^{(C)} \sim t^{-4} \quad , \quad \delta g_{rt}^{(C)} \sim t^{-5}, \\ \delta F_{tr}^{(C)} &\sim t^{-4} \quad , \quad \delta F_{t\theta}^{(C)} \sim t^{-4} \quad , \quad \delta F_{r\theta}^{(C)} \sim t^{-5}. \end{aligned}$$

D. Type D initial data

1. *Odd parity perturbations with $l \geq 2$*

$$\begin{aligned} \delta g_{t\phi}^{(D)} &\sim t^{-(2l+1)} \quad , \quad \delta g_{r\phi}^{(D)} \sim t^{-(2l+2)}, \\ \delta F_{t\phi}^{(D)} &\sim t^{-(2l+2)} \quad , \quad \delta F_{r\phi}^{(D)} \sim t^{-(2l+1)} \quad , \quad \delta F_{\theta\phi}^{(D)} \sim t^{-(2l+1)}. \end{aligned}$$

2. *Odd parity perturbations with $l = 1$*

$$\delta g_{t\phi, l=1}^{(D)} \sim t^{-5}$$

$$\delta F_{t\phi, l=1}^{(D)} \sim t^{-6} \quad , \quad \delta F_{r\phi, l=1}^{(D)} \sim t^{-5} \quad , \quad \delta F_{\theta\phi, l=1}^{(D)} \sim t^{-5}.$$

3. *Even parity perturbations with $l \geq 2$*

$$\delta g_{tt}^{(D)} \sim t^{-(2l+1)} \quad , \quad \delta g_{rr}^{(D)} \sim t^{-(2l+1)} \quad , \quad \delta g_{\theta\theta}^{(D)} \sim t^{-(2l+1)} \quad , \quad \delta g_{\phi\phi}^{(D)} \sim t^{-(2l+1)} \quad , \quad \delta g_{rt}^{(D)} \sim t^{-(2l+2)},$$

$$\delta F_{tr}^{(D)} \sim t^{-(2l+1)} \quad , \quad \delta F_{t\theta}^{(D)} \sim t^{-(2l+1)} \quad , \quad \delta F_{r\theta}^{(D)} \sim t^{-(2l+2)}.$$

4. *Even parity perturbations with $l = 1$*

$$\delta g_{tt}^{(D)} \sim t^{-5} \quad , \quad \delta g_{rr}^{(D)} \sim t^{-5} \quad , \quad \delta g_{rt}^{(D)} \sim t^{-6},$$

$$\delta F_{tr}^{(D)} \sim t^{-5} \quad , \quad \delta F_{t\theta}^{(D)} \sim t^{-5} \quad , \quad \delta F_{r\theta}^{(D)} \sim t^{-6}.$$

VII. CONCLUDING REMARKS

In this paper, we employed the results of [13] and [9] and found the late-time decay of coupled electromagnetic and gravitational perturbations outside an extremal charged black hole. In particular, we have explicitly shown that the coupled perturbations do decay (except $l = 1$ odd perturbations that might correspond to a slowly rotating Kerr-Newman black hole) and found the decay rate in a way that is consistent with Refs. [11, 13, 17]. In addition, we can notice some nontrivial features of the decay of coupled perturbations that do not appear in the scalar case. For example, we can easily see from the explicit formulas for the decay rates of the perturbations of the metric tensor and the electromagnetic field tensor (from the previous section) that for type A initial data, the quadrupole ($l = 2$) perturbations generally decay more slowly than the dipole ($l = 1$) perturbations, in contrast to the corresponding decays of scalar perturbations.⁴ For type B initial data, we get the opposite behavior, and quadrupole perturbations generally decay faster.

Coupled electromagnetic and gravitational perturbations were also investigated in [11] in the context of the horizon instability of an ERN black hole. In [11], it was shown that if the coupled perturbations and their derivatives decay outside the horizon (more specifically, that Ψ_{\pm} and their derivatives decay), then a certain linear combination of Ψ_{\pm} and its r derivatives blows up at late time on the horizon.⁵ Since we have shown that such decay of Ψ_{\pm} takes place, we may say that an instability of an ERN black hole occurs for coupled (linearized) gravitational and electromagnetic perturbations.

It would be interesting, as a future research, to try to find the full leading late-time behavior of the coupled perturbations. In order to do it, one can try to employ the so-called “late-time expansion”, presented, for example, in Refs. [19, 20], and use the exact stationary solutions given in [21]. If there are nonvanishing Aretakis or NP constants (for Ψ_{\pm}), one can also employ them for the calculation.

An additional natural extension of the current research and the one performed in [13] would be the study of Yang-Mills fields on the exterior of the ERN black hole. Such a study, for example, was carried out in [22] for the particular case of a spherically symmetric $SU(2)$ Yang-Mills field. It would be interesting to investigate it further and check whether the obtained results are related to those of [13] and the present paper.

⁴ In [10], Bicak pointed into a similar observation. However, it was based on apparently wrong results for the late-time decay of scalar perturbations.

⁵ This analysis extends the one performed by Aretakis [16] (for scalar perturbations) to coupled perturbations.

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- [1] F.J. Zerilli, *Perturbation analysis for gravitational and electromagnetic radiation in a Reissner-Nordstrom geometry*, Phys. Rev. D **9**, 860 (1974).
 - [2] V. Moncrief, *Odd-parity stability of a Reissner-Nordstrom black hole*, Phys. Rev. D **9**, 2707 (1974).
 - [3] V. Moncrief, *Stability of Reissner-Nordstrom black holes*, Phys. Rev. D **10**, 1057 (1974).
 - [4] V. Moncrief, *Gauge-invariant perturbations of Reissner-Nordstrom black holes*, Phys. Rev. D **12**, 1526 (1975).
 - [5] D. Chitre, *Perturbations of Charged Black Holes*, Phys. Rev. D **13**, 2713 (1976).
 - [6] C.H. Lee, *Coupled gravitational and electromagnetic perturbations around a charged black hole*, J. Math. Phys. **17**, 1226 (1976).
 - [7] C.H. Lee and W. D. McGlinn, *Uniqueness of perturbation of a Reissner-Nordstrom black hole*, J. Math. Phys. **17**, 2159 (1976).
 - [8] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford, UK, Clarendon, 1985).
 - [9] J. Bicak, *On the theories of the interacting perturbations of the Reissner-Nordstrom black hole*, Czech. J. Phys. B **29**, 945 (1979).
 - [10] J. Bicak, *Gravitational collapse with charge and small asymmetries. II. Interacting electromagnetic and gravitational perturbations*, Gen. Relativ. Gravit. **12**, 195 (1980).
 - [11] J. Lucietti, K. Murata, H. S. Reall, and N. Tanahashi, *On the horizon instability of an extreme Reissner-Nordstrom black hole*, J. High Energy Phys. **03** (2013) 035.
 - [12] J. Bicak, *Gravitational collapse with charge and small asymmetries. I. Scalar perturbations*, Gen. Relativ. Gravit. **3**, 331 (1972).
 - [13] O. Sela, *Late-time decay of perturbations outside extremal charged black hole*, Phys. Rev. D **93**, 024054 (2016).
 - [14] R. H. Price, *Nonspherical perturbations of relativistic gravitational collapse. I. Scalar and gravitational perturbations*, Phys. Rev. D **5**, 2419 (1972).
 - [15] R. H. Price, *Nonspherical perturbations of relativistic gravitational collapse. II. Integer-spin, zero-rest-mass fields*, Phys. Rev. D **5**, 2439 (1972).
 - [16] S. Aretakis, *Stability and instability of extreme Reissner-Nordstrom black hole spacetimes for linear scalar perturbations II*, Ann. Henri Poincaré **12**, 1491 (2011) and additional references therein.
 - [17] C. J. Blaksley and L. M. Burko, *The late-time tails in the Reissner-Nordstrom spacetime revisited*, Phys. Rev. D **76**, 104035 (2007).
 - [18] L. Barack, *Late time dynamics of scalar perturbations outside black holes. I. A shell toy-model*, Phys. Rev. D **59**, 044016 (1999).
 - [19] L. Barack, *Late time dynamics of scalar perturbations outside black holes. II. Schwarzschild geometry*, Phys. Rev. D **59**, 044017 (1999).
 - [20] A. Ori, *Transmission and reflection coefficients for a scalar field inside a charged black hole*, Phys. Rev. D **57**, 2621 (1998).
 - [21] J. Bicak, *Stationary interacting fields around an extreme Reissner-Nordstrom black hole*, Phys. Lett **64A**, 279 (1977).
 - [22] P. Bizon and M. Kahl, *A Yang-Mills field on the extremal Reissner-Nordstrom black hole*, Classical Quantum Gravity **33**, 175013 (2016).